Research Statement

John Villavert

1 Introduction

The primary focus of my research encompasses topics from nonlinear analysis, the calculus of variations, and nonlinear integral and partial differential equations (PDEs). Essentially, my projects involve the analysis of nonlinear elliptic systems, hyperbolic conservations laws, and nonlinear parabolic systems. These research projects can be classified into three categories:

- non-linear elliptic systems and fully nonlinear systems of integral equations,
- the Keller–Segel model and the incompressible Navier-Stokes equations, and
- hyperbolic conservation laws and their regularization.

The three categories above more or less constitute the research problems I have studied since my doctoral studies at the University of Colorado and as a faculty member at the University of Oklahoma (postdoc) and the University of Texas-Rio Grande Valley (tenure-track). In many of the problems, I explore notions of well-posedness, ill-posedness, and the qualitative properties of solutions for various PDEs, e.g., I examine the symmetry, monotonicity, regularity and asymptotic properties of their solutions. In fact, a number of these problems are closely related to some well-known open questions. However, the actual questions examined are often far more general and the techniques adopted or developed in their study are at times non-standard. The advantage of this is two-fold. First, it allows us to re-examine classical problems in new directions, which may lead to alternative proofs and solutions. Second, it may shed light on overcoming the difficulties faced when studying such conjectures, some of which are discussed below in greater detail.

In what follows, we introduce and motivate my research projects in more detail by describing the problems, their methods of study (and subsequently my contributions) and the possible future directions.

2 Nonlinear elliptic and related problems

The first category of research is on the analysis of nonlinear elliptic systems and related problems through the development of novel mathematical methods. Namely, I study the quantitative and qualitative properties of solutions to well-known families of integral equations and their closely related family of partial differential equations. The motivating examples within this family include many cases which correspond to fundamental problems in mathematical physics, conformal geometry, singular integral operators, and nonlinear differential equations. As a result, some important consequences of the research problems considered are closely related to the Hénon–Lane–Emden conjecture along with a new direction for generalizing the Yamabe and prescribing curvature problems in the framework of integral operators.

In order to motivate this project, let us recall some very simple examples which illustrate the same underlying principles and difficulties encountered in the problems of this research. Particularly, the prototypical example is the weighted elliptic equation commonly referred to as the Hénon–Lane–Emden equation:

$$-\Delta u = |x|^{\sigma} u^{p},\ u > 0,\ x \in \mathbb{R}^{n}, \tag{2.1}$$

where $n \geq 3$, $p > 1$ and $\sigma > -2$. This elliptic problem arises as a model for rotating stellar clusters, and when $\sigma = 0$ and $p = (n+2)/(n-2)$, the classification of solutions for this problem is also an important ingredient in the celebrated Yamabe and prescribing scalar curvature problems. For instance, if $M = \mathbb{S}^n$ with the induced metric g_0, one may ask if there exists a metric g pointwise conformal to g_0 with constant scalar curvature. It turns out this question is more or less equivalent to finding a positive solution to the semilinear equation. This problem is also connected with the critical case of the Sobolev inequality in which compactness of the embedding fails.

In recent decades, another fashionable problem centered on finding optimal criteria for the existence and non-existence of positive solutions of equation (2.1). For instance, if $\sigma = 0$, the celebrated result of Gidas and Spruck [4, 21] (see also [9, 10, 39]) established the following.

- Equation (2.1) has no solution in the subcritical case $1 < p < p_S := \frac{n+2}{n-2}$;
- it does admit positive solutions in the critical (=) and super-critical (>) cases: $p \geq \frac{n+2}{n-2}$.

Basically, this shows that the exponent $p = p_S$ is optimal and provides a complete dichotomy between the existence and non-existence of solutions. On the other hand, non-existence theorems for equation (2.1) and related problems, sometimes referred to as Liouville type theorems, are also important in establishing a priori estimates and singularity and regularity properties of solutions for a large class of elliptic problems (see [22, 49]). Another important and closely related issue is on the asymptotic behavior of solutions for equation (2.1) and we shall elucidate its connection with the Liouville type properties in greater detail shortly. For instance, since $p > \frac{n+\sigma}{n-2}$ is a necessary condition for existence (and thus $n - 2 > \frac{2+\sigma}{p-1}$), the authors in [37, 38] determined that all bound states (i.e., bounded and decaying positive solutions) of equation (2.1) vanish at infinity with either the slow rate or the fast rate, respectively:

$$u(x) \simeq |x|^{-\frac{2+\sigma}{p-1}} \ \text{ or } \ u(x) \simeq |x|^{-(n-2)}.$$

Here, the notation $f(x) \simeq g(x)$ means there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ as $|x| \longrightarrow \infty$. A natural generalization of the Hénon–Lane–Emden equation is the Hénon–Lane–Emden system:

$$\begin{cases} -\Delta u = |x|^{\sigma_1} v^{q}, & x \in \mathbb{R}^n \backslash \{0\}, \\ -\Delta v = |x|^{\sigma_2} u^{p}, & x \in \mathbb{R}^n \backslash \{0\}, \end{cases} \tag{2.2}$$

where $\sigma_i > -2$ and $p, q > 1$. Surprisingly, the qualitative properties described earlier for equation (2.1) become rather difficult to extend to system (2.2). For example, the Hénon–Lane–Emden conjecture states that the Sobolev hyperbola,

$$\frac{n+\sigma_1}{1+q} + \frac{n+\sigma_2}{1+p} = n - 2,$$

is the critical case analogue for the system. Namely, it states that system (2.2) admits no positive solution if the subcritical case holds, i.e.,

$$\frac{n+\sigma_1}{1+q} + \frac{n+\sigma_2}{1+p} > n - 2.$$

Even in the unweighted case $\sigma_i = 0$, this is often referred to as the Lane–Emden conjecture and it too has only partial results. Particularly, it holds true for radial solutions (see [45]) and for low dimension $n \le 4$ (see [49, 52, 54]). For the weighted case $\sigma_i \neq 0$, similar partial results can be found in [18, 47, 60].

We extend this study to more general poly-harmonic and integral versions of the Hénon–Lane–Emden problems. Namely, we shall consider the psuedo-differential system

$$\begin{cases} (-\Delta)^{\alpha/2} u = |x|^{\sigma_1} v^q, & x \in \mathbb{R}^n \backslash \{0\}, \\ (-\Delta)^{\alpha/2} v = |x|^{\sigma_2} u^p, & x \in \mathbb{R}^n \backslash \{0\}, \end{cases} \tag{2.3}$$

and the more general integral system involving the Riesz potentials

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^{\sigma_1} v^q(y)}{|x-y|^{n-\alpha}}\, dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^{\sigma_2} u^p(y)}{|x-y|^{n-\alpha}}\, dy, \end{cases} \tag{2.4}$$

where $n \ge 3$, $p, q > 0$, $\alpha \in (0, n)$ and $\sigma_i \in (-\alpha, \infty)$. Indeed, the close relationship between these two systems can be best illustrated by our result in [60]: Let $\alpha = 2k$ be an even integer and $p, q > 1$. Then a positive solution $u, v \in C^{2k}(\mathbb{R}^n \backslash \{0\}) \cap C(\mathbb{R}^n)$ of system (2.3), multiplied by a suitable constant if necessary, is a solution of system (2.4); and vice versa.

The motivation for studying the qualitative properties of solutions for the general integral systems stem from several problems. For example, one problem originates from the Hardy–Littlewood–Sobolev (HLS) inequality [56] (cf. [34, 40]). Namely, the problem concerns finding the best constant in the inequality, and this entails maximizing a certain functional under suitable constraints and thus requires delicate variational techniques. Interestingly, the critical points for this functional are precisely the positive solutions of system (2.4) when $\sigma_i = 0$ and

$$\frac{1}{1+q} + \frac{1}{1+p} = \frac{n-\alpha}{n}.$$

In other words, this integral system comprises of the Euler–Lagrange equations for the associated functional. On the other hand, if $\sigma_1 = \sigma_2 \doteq \sigma$, $p = q = \frac{n+\alpha-2\sigma}{n-\alpha}$ and $u \equiv v$, system (2.4) reduces to the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha} |y|^{\sigma}}\, dy. \tag{2.5}$$

In the case where $\alpha = 2$, (2.5) becomes the Euler–Lagrange equation for the classical Hardy–Sobolev inequality, which is itself a special case of the Caffarelli–Kohn–Nirenberg inequality (cf. [1, 6, 8, 12]). In view of this, we call systems (2.3) and (2.4) the Hardy–Sobolev type systems hereafter.

Main problems

This section describes the research problems in this project including partial results and their methods of study. Although the problems below include challenging open conjectures, we believe the project will, at the very least, generate more ideas and make significant contributions within the area or perhaps find applications to other significant problems. In addition, these problems will extend considerably many classical results for nonlinear elliptic equations and systems (cf. [3, 7, 17, 20, 33, 44, 45, 48, 51, 52, 53, 55] and the references therein)

Generalized Hénon–Lane–Emden conjecture

The first component extends the results for the scalar problem to the the Hardy–Sobolev type systems. In particular, we consider a generalized version of the Hénon–Lane–Emden conjecture and study the decay properties of solutions.

First, let

$$H(p,q,\sigma_i) := \frac{n+\sigma_1}{1+q} + \frac{n+\sigma_2}{1+p}.$$

Then the Hardy–Sobolev systems are said to be in the subcritical, critical, or super-critical case if $H(p,q,\sigma_i) > n-\alpha$, $H(p,q,\sigma_i) = n-\alpha$, or $H(p,q,\sigma_i) < n-\alpha$, respectively.

Conjecture 1. *(i) System* (2.4) *has no positive solution if the sub-critical case holds.*

(ii) System (2.4) *admits a positive solution if either the critical or super-critical case holds.*

So far, we have made significant progress on both aspects of Conjecture 1. Specifically, in addressing part (i) of the conjecture, we proved in [57] the existence of positive solutions for both the critical and super-critical cases provided that $\alpha = 2k$ and $\sigma_i \in (-2,\infty)$. We did this by developing an elegant degree theoretic approach for the shooting method along with a careful construction of a particular continuous “target” map. Indeed, this construction is crucial in circumventing the issues with the poly-harmonic operators $(-\Delta)^k$, which do not arise when dealing with, say, system (2.2). Then, the desired existence result follows from these ideas combined with some Pohozaev type identities. We have further developed this method and established various existence results to far general elliptic equations and systems having critical and supercritical growth (see [35, 36]).

In regards to part (ii) of the conjecture, we have resolved the conjecture for bound states, which also includes the radial solutions. Here, the solution u,v is said to be decaying if $u(x) \simeq |x|^{-\theta_1}$ and $v(x) \simeq |x|^{-\theta_2}$ for some rates $\theta_1,\theta_2 > 0$. Specifically, the non-existence of bounded and decaying solutions for (2.4) in the subcritical case was established in [59].

Decaying properties of solutions

The second component of this project concerns the decay properties of solutions for the Hardy–Sobolev systems and connects it with Conjecture 1. First, let us define the two principle rates of decay.

Definition. *Let u,v be positive solutions of system* (2.4)*. Then u,v are said to decay with the* ***slow rates*** *as $|x| \longrightarrow \infty$ if $u(x) \simeq |x|^{-q_0}$ and $v(x) \simeq |x|^{-p_0}$, where*

$$p_0 = \frac{\alpha(p+1)+(\sigma_2 p+\sigma_1)}{pq-1} \quad \textit{and} \quad q_0 = \frac{\alpha(q+1)+(\sigma_1 q+\sigma_2)}{pq-1}.$$

Without loss of generality, suppose $q \ge p$ and $\sigma_1 \le \sigma_2$. Then u,v are said to decay with the ***fast rates*** *as $|x| \longrightarrow \infty$ if*

$$u(x) \simeq |x|^{-(n-\alpha)} \quad \textit{and}$$

$$\begin{cases} v(x) \simeq |x|^{-(n-\alpha)}, & \textit{if } p(n-\alpha)-\sigma_2 > n; \\ v(x) \simeq |x|^{-(n-\alpha)} \ln|x|, & \textit{if } p(n-\alpha)-\sigma_2 = n; \\ v(x) \simeq |x|^{-(p(n-\alpha)-(\alpha-\sigma_2))}, & \textit{if } p(n-\alpha)-\sigma_2 < n. \end{cases}$$

We have shown in [59] that positive bound states of system (2.4) vanish at infinity with either the slow rates or fast rates just as in the scalar case. A key ingredient of this result uses the fact that the bounded and fast decaying solutions can be completely characterized by an integrability condition. More precisely, the bounded and fast decay solutions are indeed equivalent to the integrable solutions, i.e., $(u,v) \in L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$ where $r_0 = n/q_0$ and $s_0 = n/p_0$. Further, it was shown that integrable solutions are radially symmetric and decreasing about some point and thus, Pohozaev type identities in integral form imply that there are no positive integrable solutions in the subcritical case. Additionally, we also proved that if u, v are not integrable but are bound states, then they necessarily decay with the slow rates and so, as another consequence of Pohozaev type identities, there are no bound states in the subcritical case. In view of these results, we consider the more general statement.

Conjecture 2. *Every bounded positive solution of system* (2.4) *either decays with the slow rates or the fast rates. In particular, every bounded non-integrable solution necessarily decays with the slow rates.*

In view of the results described earlier, if Conjecture 2 were to hold, then the non-existence of solutions in the subcritical case will follow thereby providing a resolution of Conjecture 1.

A refined approach of Serrin and Zou and the Lane-Emden conjecture

Let us consider some noteworthy cases of the HLS systems and describe our recent work on obtaining Liouville theorems to elliptic problems related to a renowned conjecture. First, recall the celebrated Liouville type theorem for the Lane-Emden equation,

$$\Delta u + u^p = 0 \text{ in } \Omega \subseteq \mathbb{R}^n. \tag{2.6}$$

Theorem 1. *Let* $n \geq 3$, $\Omega = \mathbb{R}^n$ *and* $1 < p < p_S$. *If* u *is a non-negative solution of equation* (2.6), *then necessarily* $u \equiv 0$.

A natural question is if one may extend this result for the corresponding elliptic system called the Lane Emden system: $p, q > 0$ and

$$\begin{aligned} \Delta u + v^p = 0, \ \ x \in \Omega, \\ \Delta v + u^q = 0, \ \ x \in \Omega. \end{aligned} \tag{2.7}$$

Obtaining an analogous non-existence theorem for the system, however, is rather difficult. It remains an open problem and is often called the Lane-Emden conjecture. More precisely, the conjecture infers that system (2.7) has no positive classical solution in $\Omega = \mathbb{R}^n$, if and only if the subcritical condition

$$\frac{1}{1+p} + \frac{1}{1+q} > 1 - \frac{2}{n}$$

holds. Unlike Theorem 1, in which the method of moving planes provides an elegant and elementary proof, the usual suspects that worked for equation (2.6) are no longer sufficient to prove the Lane-Emden conjecture. Similarly, the method of moving planes no longer applies to even slight variants of equation (2.6) having variable coefficients with suitable growth. So far, the best possible strategy in proving the Lane-Emden conjecture centers on an approach originated by Serrin and Zou [52]; however, their method has a bottleneck in that it only works in low spatial dimensions. Thus, their method has been used to prove the Lane-Emden conjecture for dimension $n \leq 4$ [54]. Interestingly, Serrin and Zou's approach can also be applied to equation (2.6) to prove Theorem 1 but it only works for $n \leq 4$ as well. Motivated by this observation, we have recently improved the method to effectively remove the restriction on the dimension—at least we have refined the approach to give an alternative proof of Theorem 1 in all dimensions [61]. In doing so, we are able to identify the underlying obstructions inherent in Serrin and Zou's original method, then we demonstrate how to overcome them. Indeed, the merit of this work is it sheds more light on how to possibly overcome the major difficulties in the Lane-Emden conjecture.

For completeness, we give a short description of the key ideas in our new proof of Theorem 1. Roughly speaking, a key ingredient in our proof of the Liouville type theorem relies on the Rellich-Pohozaev identity: for a non-negative solution u of the Lane-Emden equation in $\Omega = B_R(0)$, there holds

$$\Big(\frac{n}{p+1} - \frac{n-2}{2}\Big) \int_{B_R(0)} u^{p+1}\, dx$$
$$= \int_{\partial B_R(0)} \Big\{ R\frac{u^{p+1}}{p+1} + R^{-1}|x\cdot Du|^2 - \frac{R}{2}|Du|^2 + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \Big\}\, dS,$$

where $\nu = x/|x|$ is the outward normal unit vector at $x \in \partial B_1(0)$ and directional derivative $\partial u/\partial \nu = \nabla u \cdot \nu$. The key is to exploit this identity to estimate the energy quantity

$$F(R) := \int_{B_R(0)} u^{p+1}\, dx,$$

since the Rellich-Pohozaev identity implies the estimate

$$F(R) \le C(G_1(R) + G_2(R)),$$

where

$$G_1(R) := R^N \int_{\mathbb{S}^{N-1}} u(R,\theta)^{p+1}\, d\theta,$$

and

$$G_2(R) := R^N \int_{\mathbb{S}^{N-1}} \Big(|Du(R,\theta)|^2 + R^{-2} u(R,\theta)^2 \Big)\, d\theta.$$

Here we are writing $u = u(r,\theta)$ in spherical coordinates with $r = |x|$ and $x/|x| \in \mathbb{S}^{N-1}$ $(x \neq 0)$, and $D = D_x$ is the gradient operator in terms of the spatial variable x. It will suffice to prove that equation (2.6) has no positive entire solutions. Therefore, if we assume u is indeed a positive solution, then we shall control the surface integrals above in terms of $F(R)$ to arrive at the feedback estimate $F(R) \le C[G_1(R) + G_2(R)] \le CR^{-a} F(R)^{1-b}$ for some $a, b > 0$. Hence, after taking a sequence $R = R_j \to \infty$, we may conclude that $\|u\|_{L^{p+1}(\mathbb{R}^N)} = 0$. Thus, $u \equiv 0$, and we arrive at a contradiction.

Basically, the advantage of Serrin and Zou's approach is that we effectively remove one degree of freedom in the spatial dimension when estimating the energy quantity. As we already mentioned, the preceding argument has a bottleneck in the sense that a weaker integrability of u is still required for the procedure to run smoothly. By weaker we mean that u is not assumed to have finite energy or $u \in L^{p+1}(\mathbb{R}^n)$, for example. This is due to certain standard estimates employed, e.g., the Sobolev and interpolation inequalities, which are sensitive to the dimension and are not strong enough to get the feedback estimates for larger N. In refining the above argument, we discover that the proof requires the existence of a number $q_0 > (n-1)(p-1)/2$ such that

$$\int_{B_1(0)} u^{q_0}\, dx \le C(n,p), \tag{2.8}$$

where $C(n,p) > 0$ depends only on n and p. Of course, elementary elliptic theory readily guarantees this integral estimate holds for $q_0 = p$ and $n \ge 2$, but the condition $q_0 = p > (n-1)(p-1)/2$ is only satisfied if $n \le 4$, and this is precisely where the restriction on n appears. Therefore, we are able to remove the obstruction on n once we obtain the local integral estimate (2.8).

Future direction: Integral systems of the Wolff type

The weighted integral systems involving Riesz potentials can be extended to include more general potentials. For example, we can establish analogous qualitative properties for the integral system containing Hardy weights and Wolff potentials:

$$\begin{cases} u(x) = c_1(x)W_{\beta,\gamma}(|y|^{\sigma_1}v^q)(x), \\ v(x) = c_2(x)W_{\beta,\gamma}(|y|^{\sigma_2}u^p)(x). \end{cases} \tag{2.9}$$

Here, the Wolff potential of a function f in $L^1_{loc}(\mathbb{R}^n)$ is defined by

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \Big(\frac{\int_{B_t(x)} f(y)\,dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\,\frac{dt}{t},$$

where $n \geq 3$, $p, q > 1$, $\gamma > 1$, $\beta > 0$ with $\beta\gamma < n$, $B_t(x) \subset \mathbb{R}^n$ denotes the ball of radius t centered at x, and the coefficients $c_1(x)$ and $c_2(x)$ are double bounded functions, i.e., there exists a positive constant $C > 0$ such that $C^{-1} \leq c_i(x) \leq C$ for all $x \in \mathbb{R}^n$. When $\beta = \alpha/2$ and $\gamma = 2$, the Wolff potential $W_{\beta,\gamma}(\cdot)$ is equivalent—up to a multiplicative positive constant—to the Riesz potential. Therefore, system (2.9) includes the Hardy–Sobolev type systems as special cases.

In [58], we established an equivalent characterization of fast decaying ground states similar to our earlier result for the Hardy–Sobolev type systems. Consequently, we also obtained a characterization of the fast decaying weak solutions for quasilinear systems e.g. systems involving p-Laplacians. Hence, as indicated by these results, we would like to know whether or not the other qualitative results for system (2.4) can be extended to system (2.9), and this is one potential direction of this research. For recent papers closely related to this research plan, see [14, 62].

Future direction: Prescribing integral curvature equations

As suggested in our results for integral systems, we strongly endorse this approach of adopting integral operators since it has a number of advantages. On the one hand, the ideas and methods from this approach will undoubtedly generate new directions in approaching classical problems, which may perhaps lead to new insights and resolutions to some open conjectures, including the ones stated above. On the other hand, not only will it generalize the aforementioned scalar curvature equations, it also enables one to make sense of problems beyond those involving local differential operators. To illustrate this, consider the bubbling function, $u_\epsilon(x) = (\epsilon^2 + |x|^2)/\epsilon$, for the biharmonic operator on $\mathbb{R}^2$; however, it does not satisfy $(-\Delta)^2 u = u^{-3}$, $u > 0$, in $\mathbb{R}^2$. Therefore, one can ask if there is a curvature equation in which u_ϵ associates with. Fortunately, u_ϵ is indeed a solution, modulo a positive constant, of the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^{-3}(y)}{|x-y|^2}\,dy,\ u > 0 \ \text{ in } \mathbb{R}^2.$$

Hence, for $\alpha \neq n$, a natural question we study is the solvability of the more general problem

$$u(x) = \int_M \frac{R(y)u^p(y)}{|x-y|^{n-\alpha}}\,dV_g(y),\ u > 0 \ \text{ in } M, \tag{2.10}$$

thereby formulating new curvature functions on, say, $(M, g) = (\mathbb{S}^n, g_0)$ but in terms of integral operators. Interestingly, for $\alpha > n$, existence results for (2.10) were established under antipodally symmetric $R(x)$ (see [65]). The technical ideas developed there involves new variational approaches combined with a reversed version of the HLS inequality (see [16, 15]) and many interesting questions arise as a result of these new ideas. For example, we plan to examine the qualitative properties of solutions for problems closely related to equation (2.10) but for $\alpha \neq n$ and various exponents p. Of course, we plan to study whether or not the qualitative properties for the scalar equation (2.1) can translate to this situation as well.

3 Nonlinear parabolic equations

Unlike the previous research topic, my research on nonlinear parabolic systems, in particular the Keller–Segel model, the Euler equations, and the Navier–Stokes equations, is one area which I have considered more recently. Within my research group at the University of Colorado and the University of Oklahoma, we have studied several important yet fundamental problems on the Navier–Stokes equations. This includes papers that study various existence theory for weak and mild solutions (see [27, 32]) and the study of Louiville type theorems, singularity analysis and local regularity of axi-symmetric solutions (see [29, 50]). I firmly believe that my knowledge-base from these monumental works will result in significant collaborations and results in the analysis of these fluid models.

The two-dimensional Keller–Segel model

First, let us describe our recent work on both the well-posedness and ill-posedness to a well-known chemotaxis model in two dimensions—the Keller–Segel model of the parabolic-parabolic type,

$$\partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \tag{3.1}$$

$$\partial_t v - \Delta v - u = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \tag{3.2}$$

$$(u, v)|_{t=0} = (u_0, v_0) \qquad \text{in } \mathbb{R}^2. \tag{3.3}$$

Here, $\mathbb{R}_+ := (0, \infty)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, $u = u(t, x)$ and $v = v(t, x)$ are the scalar-valued density of amoebae and the scalar-valued concentration of chemical attractant, respectively, while (u_0, v_0) are the given initial data. The term chemotaxis refers to the attraction and movement of cellular organisms such as amoebae or bacteria in response to chemical stimulation. The Keller–Segel model, first introduced by Keller and Segel in [28] (see also [11]), is perhaps the most common basic model for describing this motion of cell migration through chemical attraction.

When studying such nonlinear physical systems, there are several primary aspects of concern. One aspect is on the basic property of local or global-in-time well-posedness of the problem. We may ask if solutions exist in some sense, are they unique, and do they vary continuously upon small perturbations of the initial data. Another closely related aspect concerns the setting in which the model is ill-posed. In fact, our main result provides a thorough analytical examination of this model by identifying the proper functional space setting in terms of the Triebel–Lizorkin spaces in which the Cauchy problem is ill-posed. More specifically, we examine the dividing number with respect to r for the well-posedness of solutions with initial data (u_0, v_0) belonging to the Triebel-Lizorkin spaces $\dot{F}_2^{-1,r}(\mathbb{R}^2) \times \dot{F}_\infty^{0,2}(\mathbb{R}^2)$. Remarkably, for the two-dimensional Keller–Segel model, we proved that the dividing number is $r = 2$ (see [13]). By the dividing number we mean that well-posedness holds for $r = 2$, but the system is ill-posed whenever $2 < r \leq \infty$. As a result of establishing this relationship between well-posedness and ill-posedness, we find the optimal setting in which the model remains valid while gaining a deeper understanding of the setting in which the model fails to capture basic deterministic properties.

Remarks on well-posedness of solutions

First, recall a Cauchy problem is said to be *locally well-posed* in Z if for every initial data $u_0 \in Z$ there exists a time $T = T(u_0) > 0$ such that

(1) a solution of the initial value problem exists in the time interval $[0, T]$,

(2) is unique in a certain Banach space of functions $Y \subset C([0, T]; Z)$,

(3) the solution map from u_0 to the solution u is continuous from Z to $C([0, T); Z)$.

Furthermore, if T can be taken arbitrarily large, we say that the Cauchy problem is *globally well-posed*, and we say the Cauchy problem is *ill-posed* if it is not well-posed. By solutions of the Keller–Segel model, we mean mild solutions to the equivalent system of integral equations as follows:

$$\begin{aligned} u &= e^{t\Delta}u_0 - B(u, v), \\ v &= e^{t\Delta}v_0 + L(u), \end{aligned}$$

where

$$B(u, v) := \int_0^t e^{(t-\tau)\Delta}\nabla \cdot (u\nabla v)\, d\tau \quad \text{and} \quad L(u) := \int_0^t e^{(t-\tau)\Delta} u\, d\tau, \tag{3.4}$$

are the bilinear and linear terms, respectively. Indeed, the equations (3.1)–(3.2) are scale invariant under the transformations

$$(u(t, x), v(t, x)) \rightarrow (\lambda^2 u(\lambda^2 t, \lambda x), v(\lambda^2 t, \lambda x)) \quad \text{for all } \lambda > 0.$$

Thus, the local well-posedness is obtained through a standard fixed point argument on the integral equations in a suitable function space. One can also further exploit the scaling invariance of the equations by choosing a proper "critical" function space that preserves the scaling. Therefore, the global well-posedness of solutions naturally follows under some smallness assumption on the initial data. This idea of using a functional setting invariant by scaling is now classical and originates from several works (cf. [23, 26, 27, 30, 31, 41, 64] and the references therein). One example of a critical space for the Keller–Segel model is $\dot{F}_2^{-1,2}(\mathbb{R}^2) \times \dot{F}_\infty^{0,2}(\mathbb{R}^2)$. So, in our paper [13], we derived the linear and bilinear estimates in this critical space and applied the usual fixed point argument in order to get the local well-posedness result along with the global well-posedness for small initial data.

Norm Inflation

To show the ill-posedness of system (3.1)–(3.3), we implemented the novel framework of norm-inflation pioneered by Bourgain and Pavlović [2] in their study of the ill-posedness of the Navier–Stokes equation in the largest critical space $\dot{B}_\infty^{-1,\infty}$; but in doing so, we have contributed new ideas in our adoption of their techniques. Let us describe the general idea for showing ill-posedness via norm inflation, but first,

Our ill-posedness result showed that the third condition (3) of continuity is violated by showing the onset of norm inflation, namely, we construct a particular class of arbitrarily small initial data that produce arbitrarily large solutions in arbitrarily short time. Particularly, we demonstrate that the culprit responsible for generating norm inflation lies in the bilinear term within the model. Therefore, it is the density u in the Keller–Segel model which exhibits norm inflation. Roughly speaking, the key steps to showing this norm inflation property is to first decompose the integral system, especially the bilinear term, into several parts: one part stemming from the bilinear term responsible for norm inflation and the remaining terms which can be controlled. The a priori estimates for solutions of the Cauchy problem in $\dot{F}_2^{-1,r=2}(\mathbb{R}^2)\times BMO(\mathbb{R}^2)$ is an important ingredient in this step since they are exploited in order to control some of those remaining terms in the decomposition. The $\dot{F}_2^{-1,r>2}(\mathbb{R}^2)$–norm of the solution u in arbitrary short time can then be bounded from below by the norm inflation term and the controlled terms. Thus, this proves the solution map for u is discontinuous at the initial time. We refer the reader to [13] for the details.

Future direction: the incompressible Navier–Stokes

We shall consider the modeling and analysis of fluid flows with emphasis on the dynamic stability of the three-dimensional incompressible Euler and Navier–Stokes equations. This research is closely related to the famous open question of whether the three-dimensional Navier–Stokes equations can develop a finite-time singularity from smooth initial data [19]. The understanding of this fundamental property would enhance our knowledge of fluid dynamic stability and shed light on the onset of turbulence. To illustrate some key aspects of this research, consider the incompressible Navier–Stokes equations,

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{3.5}$$

with initial condition $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0$, where $\mathbf{u}$ is the velocity, p is the pressure, and ν is the viscosity constant. The initial condition $\mathbf{u}_0$ is assumed to be smooth, divergence–free, and has finite energy. Let $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ be the vorticity. By applying the curl operator to (3.5), one obtains the vorticity equation:

$$\boldsymbol{\omega}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\Delta\boldsymbol{\omega}, \tag{3.6}$$

with initial condition $\boldsymbol{\omega}(\mathbf{x},0)=\nabla\times\mathbf{u}_0$. The first term on the right hand side of (3.6) is called the vortex stretching term. This term is absent in the two-dimensional case. The vortex stretching term is responsible for the dynamic generation of small scales. Formally, the vortex stretching term has a quadratic nonlinearity in vorticity. In some sense, most regularity analysis treats the nonlinear terms as a small perturbation of the diffusion equation, which works only if the solution is small in some scaling invariant norm.

Due to the supercritical nature of the nonlinearity of the Navier–Stokes equations, these equations with large initial data are convection dominated, instead of diffusion dominated. For this reason, we believe that the understanding of whether the corresponding Euler equations exhibit finite-time blowup could shed useful light on the global regularity of the Navier-Stokes equations. Let us consider the Euler equations in the vorticity form. One important observation is that when we consider the convection term together with the vortex stretching term, the two nonlinear terms form a commutator or a Lie derivative:

$$\boldsymbol{\omega}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = 0. \tag{3.7}$$

It is reasonable to believe that the commutator would lead to some cancellation among the two nonlinear terms, and thus weaken the nonlinearity dynamically. This points to the potential important role of convection in the Euler equations. Another way to realize the importance of convection is to use the Lagrangian formulation of the vorticity equation:

$$\boldsymbol{\omega}(X(\alpha,t),t) = X_\alpha(\alpha,t)\boldsymbol{\omega}_0(\alpha), \tag{3.8}$$

where $X(\alpha,t)$ is the Lagrangian flow map: $X_t=\mathbf{u}(X,t)$, $X(\alpha,0)=\alpha$. Due to the incompressibility of the flow, the flow map is volume preserving, i.e., $\det(X_\alpha(\alpha,t))\equiv 1$. Thus vorticity can increase dynamically only through the dynamic deformation of the Lagrangian flow map, but under the constraint: $\det(X_\alpha(\alpha,t)) \equiv 1$. A formal asymptotic analysis suggests that as vorticity increases, the local support of maximum vorticity tends to severely deform and flatten. Such deformation tends to weaken the nonlinearity of vortex stretching dynamically.

Revealing the stabilizing effect of convection via a three-dimensional model

We propose to study the stabilizing effect of convection via a new three-dimensional model. As we shall see below, this model is derived from a reformulation of the axi-symmetric Navier-Stokes equations. The only difference between our three-dimensional model and the reformulated Navier–Stokes equations is that we drop the convection term in the model. If we add the convection term back to the model, we will recover the full Navier–Stokes equations.

Consider the three-dimensional axi-symmetric Navier-Stokes equations with swirl. Denote by u^θ, ω^θ and ψ^θ the angular velocity, angular vorticity and angular stream function respectively. In [24], Hou and Li introduced the following change of variables:

$$u_1 = u^\theta/r, \quad \omega_1 = \omega^\theta/r, \quad \psi_1 = \psi^\theta/r, \tag{3.9}$$

and derived the following reformulation of the axi-symmetric Navier-Stokes equations that governs the dynamics of u_1, ω_1 and ψ_1 as follows:

$$\begin{cases} \partial_t u_1 + u^r\partial_r u_1 + u^z\partial_z u_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)u_1 + 2u_1\psi_{1z}, \\ \partial_t \omega_1 + u^r\partial_r \omega_1 + u^z\partial_z \omega_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\omega_1 + \big(u_1^2\big)_z, \\ -\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\psi_1 = \omega_1, \end{cases} \tag{3.10}$$

where $u^r = -r\psi_{1z}, \quad u^z = 2\psi_1 + r\psi_{1r}$. The 1D model:

$$(u_1)_t + 2\psi_1(u_1)_z = \nu(u_1)_{zz} - 2v_1u_1, \tag{3.11}$$
$$(v_1)_t + 2\psi_1(v_1)_z = \nu(v_1)_{zz} + (u_1)^2 - (v_1)^2 + c(t), \tag{3.12}$$

is derived by setting $r = 0$ and neglecting the r-derivatives from (3.10). The equation (3.12) for v_1 was derived by integrating the w_1-equation with respect to z. As observed by Liu and Wang [43], if $\mathbf{u}$ is a smooth velocity field, then u^θ, ω^θ and ψ^θ must satisfy: $u^\theta\big|_{r=0} = \omega^\theta\big|_{r=0} = \psi^\theta\big|_{r=0} = 0$. Thus u_1, ψ_1 and ω_1 are well defined. We note that $u_1 \approx (u^\theta)_r$ near $r = 0$, which characterizes the radial vorticity.

The dynamic stability demonstrated through this 1D model strongly suggests that convection plays an essential role in stabilizing the vortex stretching term. To further investigate the stabilizing effect of convection for the Navier–Stokes equations, we propose the following three-dimensional model. This model is derived by simply dropping the convective term from the reformulated Navier–Stokes equations (3.10):

$$\begin{cases} \partial_t u_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)u_1 + 2u_1\psi_{1z}, \\ \partial_t \omega_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\omega_1 + (u_1^2)_z, \\ -\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\psi_1 = \omega_1. \end{cases} \tag{3.13}$$

Note that (3.13) is already a closed system. The main difference between this new three-dimensional model and the original Navier–Stokes equations is that we neglect the convection term in our model.

If we add the convection term back to our three-dimensional model, we will recover the Navier–Stokes equations.

To see how convection depletes the mechanism for generating a potential finite-time singularity of our model, we add the convection term back to the model. We use the solution of the viscous model at a time sufficiently close to the potential singularity time as the initial condition for the full 3d Navier–Stokes equations. Surprisingly, the numerical studies show that the solution of the 3d Navier–Stokes equations immediately becomes less focused and smoother along the symmetry axis. As time increases, the solution develops a thin jet that moves away from the symmetry axis. As we know from the Caffarelli-Kohn-Nirenberg partial regularity theory [5] (see also [42] for a simplified proof), the three-dimensional axisymmetric Navier–Stokes equations cannot develop finite-time singularities away from the symmetry axis. The fact that the convection term forces the most singular part of the solution to move away from the symmetry axis shows that convection has effectively destroyed the mechanism that leads to a potential finite-time blowup observed in the model. One significant application of this stabilizing effect of convection may prove useful in gaining a deeper understanding of the global regularity of the Navier–Stokes equations. Hence, the main goal of this research is to rigorously prove these analytical properties such as the finite-time blowup of solutions and convection stabilizing effect of these models. However, this is only one possible direction in a number of paths we may take in studying incompressible flows.

4 Hyperbolic conservation laws

My research on nonlinear hyperbolic problems focused on inviscid regularization methods for conservation laws with the equations of gas dynamics in mind. Instead of applying classical viscous or dispersive perturbations, a spatial-averaging is applied to the nonlinear terms of the PDEs. This notion of averaging directly stems from the successful regularization of the inviscid Burgers equation via averaging of the convective velocity [25, 46],

$$\begin{cases} u_t + \overline{u} u_x = 0 \\ \overline{u} = g^\alpha * u \\ g^\alpha(x) = \frac{1}{2\alpha} e^{-|x|/\alpha}. \end{cases} \tag{4.1}$$

An important application of such a method lies in its potential in alleviating the major difficulties faced in the numerical simulation of compressible fluid flows. However, to be a valid regularization method, the global well-posedness and stability/convergence of the regularized solutions should be verified. Recently, we have examined a generalization of this regularization to symmetric hyperbolic system of N equations in n-space variables

$$u_t + \sum_i^n A_i(x,t,u) u_{x_i} = h(x,t,u) \quad \text{in } U_T = \mathbb{R}^n \times (0,T), \tag{4.2}$$

where the A_i's are symmetric $N \times N$ matrices while h, u are N-vector-valued functions. We always prescribe an initial condition to this system

$$u(x,0) = u_0(x). \tag{4.3}$$

In our manuscript [63], we introduced spatial averaging to the coefficient matrices in (4.2) to prevent the finite-time blowup of solutions, thus obtaining global well-posedness of smooth solutions to this modified IVP. More precisely, we filter (4.2) into the following system:

$$\begin{cases} u_t + \sum_i^n \overline{A_i(x,t,u)} u_{x_i} = h(x,t,u) \quad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), \end{cases} \tag{4.4}$$

where $\overline{A}$ is the convolution product of the entries of A with respect to the x-variables with some specifically chosen kernel g. One example is the Helmholtz filter which was used in (4.1). We then proved the global-in-time existence and uniqueness of classical solutions of this averaged IVP via standard Sobolev energy estimates. Moreover, we showed that this regularization captures the behavior of the solutions to the 1D Cauchy problem for systems of conservation laws. In particular, we proved via a BV compactness argument, that the regularized solutions will converge, as the level of filtering vanishes, to a weak solution to the original, non-averaged system. Moreover, under more stringent conditions on the filter and initial data, this limiting weak solution is actually the unique entropy admissible solution.

References

[1] M. Badiale and G. Tarantello. A Hardy–Sobolev inequality with applications to a nonlinear elliptic equation arising in astrophysics. *Arch. Ration. Mech. Anal.*, 163:259–293, 2002.

[2] J. Bourgain and N. Pavlović. Ill-posedness of the Navier–Stokes equations in critical space in 3d. *J. Funct. Anal.*, 255(9):2233–2247, 2008.

[3] J. Busca and R. Manásevich. A Liouville-type theorem for Lane–Emden systems. *Indiana Univ. Math. J.*, 51:37–51, 2002.

[4] L. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42:271–297, 1989.

[5] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Comm. Pure Appl. Math.*, 35:771–831, 1982.

[6] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compositio Mathematica*, 53(3):259–275, 1984.

[7] G. Caristi, L. D'Ambrosio, and E. Mitidieri. Representation formulae for solutions to some classes of higher order systems and related Liouville theorems. *Milan J. Math.*, 76(1):27–67, 2008.

[8] F. Catrina and Z. Wang. On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2):229–258, 2001.

[9] W. Chen and C. Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63:615–622, 1991.

[10] W. Chen, C. Li, and B. Ou. Classification of solutions for an integral equation. *Comm. Pure Appl. Math.*, 59:330–343, 2006.

[11] S. Childress and J. K. Percus. Nonlinear aspects of chemotaxis. *Math. Biosci.*, 56:217–237, 1981.

[12] K.S. Chou and C. W. Chu. On the best constant for a weighted Sobolev–hardy inequality. *J. Lond. Math. Soc.*, 2:137–151, 1993.

[13] C. Deng and J. Villavert. Ill-posedness of the two-dimensional Keller–Segel model in Triebel–Lizorkin spaces. *Nonlinear Analysis*, 95:38–49, 2014.

[14] J. Dou, F. Ren, and J. Villavert. Classification of positive solutions to a Lane-Emden type integral system with negative exponents. *Discrete Contin. Dyn. Syst.*, 2016. To Appear.

[15] J. Dou and M. Zhu. Reversed Hardy-Littlewood-Sobolev inequality. *Int. Math. Res. Notices*, (19):9696–9726, 2015.

[16] J. Dou and M. Zhu. Sharp Hardy-Littlewood-Sobolev inequality on the upper half space. *Int. Math. Res. Notices*, (3):651–687, 2015.

[17] M. Fazly. Liouville theorems for the polyharmonic Henon–Lane–Emden system. *Methods Appl. Anal.*, 21(2):265–282, 2014.

[18] M. Fazly and N. Ghoussoub. On the Hénon-Lane-Emden conjecture. *Discrete Contin. Dyn. Syst.*, 34(6):2513–2533, 2014.

[19] C. Fefferman. Official clay prize problem description: Existence and smoothness of the Navier–Stokes equation, 2000.

[20] B. Gidas, W. Ni, and L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in R^n. *Adv. Math. Suppl. Studies A*, 7:369–402, 1981.

[21] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure and Appl. Math.*, 34(4):525–598, 1981.

[22] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8):883–901, 1981.

[23] Y. Guo and H. J. Hwang. Pattern formation (i): the Keller–Segel model. *J. Differential Equations*, 249:1519–1530, 2010.

[24] T. Y. Hou and C. Li. Dynamic stability of the 3D axi-symmetric Navier–Stokes equations with swirl. *CPAM*, 61(5):661–697, 2008.

[25] H. Zhao K. Mohseni and J. E. Marsden. Shock regularization for the Burgers equation. *AIAA paper 2006-1516*, 101, 2006.

[26] Y. Kagei and Y. Maekawa. On asymptotic behavior of solutions to parabolic systems modeling chemotaxis. *J. Differential Equations*, 253:2951–2992, 2012.

[27] T. Kato. Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with application to weak solutions. *Math. Z.*, 187:471–480, 1984.

[28] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415, 1970.

[29] G. Koch, N. Nadirashvili, G. Seregin, and V. Sverak. Liouville theorems for the Navier–Stokes equations and applications. *Acta Math*, 203(1):83–105, 2009.

[30] H. Kozono and Y. Sugiyama. Keller–Segel system of parabolic-parabolic type with initial data in weak $l^{n/2}$ and its application to self-similar solutions. *Indiana Univ. Math. J.*, 57:1467–1500, 2008.

[31] H. Kozono, Y. Sugiyama, and T. Wachi. Existence and uniqueness theorem on mild solutions to the Keller–Segel system in the scaling invariant space. *J. Differential Equations*, 252:1213–1228, 2012.

[32] J. Leray. Sur le mouvement d'un liquide visqueus emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[33] C. Li and J. Lim. The singularity analysis of solutions to some integral equations. *Commun. Pure Appl. Anal.*, 6(2):453–464, 2007.

[34] C. Li and J. Villavert. An extension of the Hardy–Littlewood–Pólya inequality. *Acta Math. Sci.*, 31(6):2285–2288, 2011.

[35] C. Li and J. Villavert. A degree theory framework for semilinear elliptic systems. *Proc. Amer. Math. Soc.*, 144(9):3731–3740, 2016.

[36] C. Li and J. Villavert. Existence of positive solutions to semilinear elliptic systems with supercritical growth. *Comm. Partial Differential Equations*, 41(7):1029–1039, 2016.

[37] Y. Li. Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$. *J. Differential Equations*, 95:304–330, 1992.

[38] Y. Li and W. M. Ni. On conformal scalar curvature equations in $\mathbb{R}^n$. *Duke Math. J.*, 57:895–924, 1988.

[39] Y. Y. Li and M. Zhu. Uniqueness theorems through the method of moving spheres. *Duke Math. J.*, 80(2):383–418, 1995.

[40] E. Lieb. Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. *Ann. of Math.*, 118:349–374, 1983.

[41] C. S. Lin, W. M. Ni, and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations*, 72:1–27, 1998.

[42] F.H. Lin. A new proof of the Caffarelli-Korn-Nirenberg theorem. *Comm. Pure Appl. Math.*, 51(3):241–257, 1998.

[43] J. G. Liu and W. C. Wang. Convergence analysis of the energy and helicity preserving scheme for axisymmetric flows. *SINUM*, 44:2456–2480, 2006.

[44] E. Mitidieri. A Rellich type identity and applications. *Comm. Partial Differential Equations*, 18(1–2):125–151, 1993.

[45] E. Mitidieri. Nonexistence of positive solutions of semilinear elliptic systems in R^N. *Differ. Integral Equations*, 9:465–480, 1996.

[46] G. Norgard and K. Mohseni. A regularization of the Burgers equation using a filtered convective velocity. *J. Phys. A: Math. Theor.*, 41:1–21, 2008.

[47] Q. H. Phan. Liouville-type theorems and bounds of solutions for Hardy–Hénon systems. *Adv. Differential Equations*, 17(7-8):605–634, 2012.

[48] Q. H. Phan and P. Souplet. Liouville-type theorems and bounds of solutions of Hardy–Hénon equations. *J. Differential Equations*, 252:2544–2562, 2012.

[49] P. Poláčik, P. Quittner, and P. Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems. *Duke Math. J.*, 139(3):555–579, 2007.

[50] G. Seregin and V. Sverak. On type I singularities of the local axi-symmetric solutions of the Navier–Stokes equations. *Comm. Partial Differential Equations*, 34(1-3):171–201, 2009.

[51] J. Serrin and H. Zou. Non-existence of positive solutions of semilinear elliptic systems. *Discourses in Mathematics and its Applications*, 3:55–68, 1994.

[52] J. Serrin and H. Zou. Non-existence of positive solutions of Lane–Emden systems. *Differ. Integral Equations*, 9(4):635–653, 1996.

[53] J. Serrin and H. Zou. Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities. *Acta Mathematica*, 189(1):79–142, 2002.

[54] P. Souplet. The proof of the Lane–Emden conjecture in four space dimensions. *Adv. Math.*, 221(5):1409–1427, 2009.

[55] M. A. S. Souto. A priori estimates and existence of positive solutions of nonlinear cooperative elliptic systems. *Differ. Integral Equations*, 8:1245–1245, 1995.

[56] E. B. Stein and G. Weiss. Fractional integrals on n-dimensional *Euclidean* space. *J. Math. Mech.*, 7(4):503–514, 1958.

[57] J. Villavert. Shooting with degree theory: Analysis of some weighted poly-harmonic systems. *J. Differential Equations*, 257(4):1148–1167, 2014.

[58] J. Villavert. A characterization of fast decaying solutions for quasilinear and Wolff type systems with singular coefficients. *J. Math. Anal. Appl.*, 424(2):1348–1373, 2015.

[59] J. Villavert. Qualitative properties of solutions for an integral system related to the Hardy–Sobolev inequality. *J. Differential Equations*, 258(5):1685–1714, 2015.

[60] J. Villavert. Sharp existence criteria for positive solutions of Hardy–Sobolev type systems. *Commun. Pure Appl. Anal.*, 14(2):493–515, 2015.

[61] J. Villavert. A refined approach for non-negative entire solutions of $\Delta u + u^p = 0$ under subcritical Sobolev growth . *Preprint*, 2016.

[62] J. Villavert. Asymptotic and optimal Liouville properties for Wolff type integral systems. *Nonlinear Anal.*, 130:102–120, 2016.

[63] J. Villavert and K. Mohseni. An inviscid regularization of hyperbolic conservation laws. *J. Appl. Math. Comput.*, 43(1-2):55–73, 2013.

[64] M. Winkler. Aggresgation vs. global diffusive behavior in the higher-dimensional Keller–Segel model. *J. Differential Equations*, 248:2889–2905, 2010.

[65] M. Zhu. Prescribing integral curvature equation. *Differential Integral Equations*, 29:9–10, 2016.